A curious polynomial interpolation of Carlitz-Riordan's q-ballot numbers

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A CURIOUS POLYNOMIAL INTERPOLATION OF CARLITZ-RIORDAN'S q-BALLOT NUMBERS

FRÉDÉRIC CHAPOTON AND JIANG ZENG

Abstract. We study a polynomial sequence $C_n(x|q)$ defined as a solution of a q-difference equation. This sequence, evaluated at q-integers, interpolates Carlitz-Riordan's q-ballot numbers. In the basis given by some kind of q-binomial coefficients, the coefficients are again some q-ballot numbers. We obtain in a combinatorial way another curious recurrence relation for these polynomials.

1. Introduction

This paper was motivated by a previous work of the first author on flows on rooted trees [Cha13b], where the well-known Catalan numbers and the closely related ballot numbers played an important role. In fact, one can easily introduce one more parameter q in this work, and then Catalan numbers and ballots numbers get replaced by their q-analogues introduced a long time ago by Carlitz-Riordan [CR64], see also [Car72, FH85].

These q-Catalan numbers have been recently considered by many people, see for example [Cig05, BF07, Hag08, BP12], including some work by Reineke [Rein05] on moduli space of quiver representations.

Inspired by an analogy with another work of the first author on rooted trees [Cha13a], it is natural to try to interpolate the q-ballot numbers. In the present article, we prove that this is possible and study the interpolating polynomials.

Our main object of study is a sequence of polynomials in x with coefficients in $\mathbb{Q}(q)$, defined by the q-difference equation:

$$\Delta_q C_{n+1}(x|q) = qC_n(q^2x + q + 1|q), \tag{1.1}$$

where $\Delta_q f(x) = (f(1 + qx) - f(x))/(1 + (q - 1)x)$ is the Hahn operator.

After reading a previous version of this paper, Johann Cigler has kindly brought the two related references [Cig97, Cig98] to our attention, where a sequence of more general polynomials $G_n(x, r)$ was introduced through a q-difference operator for q-integer x and positive integer r. Comparing these two sequences one has

$$G_n(qx + 1, 2) = C_{n+1}(x|q).$$

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In the next section we recall classical material on Carlitz-Riordan's q-analogue for Catalan and ballot numbers and define our polynomials. In the third section, we evaluate our polynomials at q-integers in terms of q-ballot numbers and prove a product formula when $q = 1$. In the fourth section, we find their expansion in a basis made of a kind of q-binomial coefficients and obtain another recurrence for these polynomials. This recurrence is not usual even in the special $x = q = 1$ case and we have only a combinatorial proof in the general case. We conclude the paper with some open problems.

Nota Bene: Figures are best viewed in color.

2. Carlitz-Riordan's q-ballot numbers

Recall that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ may be defined as solutions to

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad (n \geq 0), \quad C_0 = 1. \tag{2.1}$$

The first values are

n	0	1	2	3	4	5	6	7	8
C_n	1	1	2	5	14	42	132	429	1430

It is well known that C_n is the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$, which do not pass above the line $y = x$. As a natural generalization, one considers the set $\mathcal{P}(n, k)$ of lattice paths from $(0, 0)$ to $(n + 1, k)$ with steps $(1, 0)$ and $(0, 1)$, such that the last step is $(1, 0)$ and they never rise above the line $y = x$. Let $f(n, k)$ be the cardinality of $\mathcal{P}(n, k)$. The first values of $f(n, k)$ are given in Table 1. These numbers are called ballot numbers and have a long history in the literature of combinatorial theory. Moreover, one (see [Com74]) has the explicit formula

$$f(n, k) = \frac{n - k + 1}{n + 1}\binom{n + k}{k} \qquad (n \geq k \geq 0). \tag{2.2}$$

Carlitz and Riordan [CR64] introduced the following q-analogue of these numbers

$$f(n, k|q) = \sum_{\gamma \in \mathcal{P}(n,k)} q^{A(\gamma)}, \tag{2.3}$$

where $A(\gamma)$ is the area under the path (and above the x-axis). The first values of $f(n, k|q)$ are given in Table 2. Furthermore, Carlitz [Car72] uses a variety of elegant techniques to derive several basic properties of the $f(n, k|q)$, among which the following is the basic recurrence relation

$$f(n, k|q) = qf(n, k - 1|q) + q^k f(n - 1, k|q) \qquad (n, k \geq 0), \tag{2.4}$$

n\k	0	1	2	3	4	5	6
0	**1**						
1	1	**1**					
2	**1**	2	**2**				
3	1	**3**	5	**5**			
4	**1**	4	**9**	14	**14**		
5	1	**5**	14	**28**	42	**42**	
6	**1**	6	**20**	48	**90**	132	**132**

TABLE 1. The first values of ballot numbers $f(n,k)$

n\k	0	1	2	3	4
0	1				
1	1	q			
2	1	$q+q^2$	q^2+q^3		
3	1	$q+q^2+q^3$	$q^2+q^3+2q^4+q^5$	$q^3+q^4+2q^5+q^6$	
4	1	$q+q^2+q^3+q^4$	$q^2+q^3+2q^4+2q^5+2q^6+q^7$	q^3Y	q^4Y

TABLE 2. The first values of q-ballot numbers $f(n,k|q)$ with $Y = q^6+3q^5+3q^4+3q^3+2q^2+q+1$

where $f(n,k|q)=0$ if $n<k$ and $f(0,0|q)=1$.

It is also easy to see that the polynomial $f(n,k|q)$ is of degree $kn-k(k-1)/2$ and satisfies the equation $f(n,n|q)=qf(n,n-1|q)$. If one defines the q-Catalan numbers by

$$C_{n+1}(q)=\sum_{k=0}^{n} f(n,k|q)=q^{-n-1}f(n+1,n+1|q) \qquad (n\geq 0), \tag{2.5}$$

then, one obtains the following analogue of (2.1) for the Catalan numbers

$$C_{n+1}(q)=\sum_{i=0}^{n} C_i(q)C_{n-i}(q)q^{(i+1)(n-i)}, \tag{2.6}$$

where $C_0(q)=1$. Setting $\widetilde{C}_n(q)=q^{\binom{n}{2}}C_n(q^{-1})$, one has a simpler q-analog of (2.1)

$$\widetilde{C}_{n+1}(q)=\sum_{i=0}^{n} q^i\widetilde{C}_i(q)\widetilde{C}_{n-i}(q). \tag{2.7}$$

The first values are $C_1(q) = 1$, $C_2(q) = 1 + q$, $C_3(q) = 1 + q + 2q^2 + q^3$ and

$$C_4(q) = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6.$$

No explicit formula is known for Carlitz-Riordan's q-Catalan numbers. However, Andrews [And75] proved the following recurrence formula

$$C_n(q) = \frac{q^n}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q + q\sum_{j=0}^{n-1}(1-q^{n-j})q^{(n+1-j)j}\begin{bmatrix}2j+1\\ j\end{bmatrix}_q C_{n-1-j}(q), \tag{2.8}$$

where $[x]_q = \frac{q^x-1}{q-1}$.

Recall that the q-shifted factorial $(x;q)_n$ is defined by

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad (n \geq 1) \quad \text{and} \quad (x;q)_0 = 1.$$

The two kinds of q-binomial coefficients are defined by

$$\begin{bmatrix}n\\ k\end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad \binom{x}{k}_q := \frac{x(x-1)\dots(x-[k-1]_q)}{[k]_q!},$$

with $\binom{x}{0}_q = 1$. Note that

$$\binom{[n]_q}{k}_q = q^{\binom{k}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_q, \qquad \binom{[-n]_q}{k}_q = (-1)^k q^{-kn}\begin{bmatrix}k+n-1\\ k\end{bmatrix}_q.$$

The q-derivative operator $\mathcal{D}_q$ and Hahn operator Δ_q are defined by

$$\mathcal{D}_q f(x) = \frac{f(qx)-f(x)}{(q-1)x} \quad \text{and} \quad \Delta_q f(x) = \frac{f(1+qx)-f(x)}{1+(q-1)x}. \tag{2.9}$$

Definition 1. The sequence of polynomials $\{C_n(x|q)\}_{n\geq 1}$ is defined by the q-difference equation (1.1) or equivalently

$$\frac{C_{n+1}(x|q) - C_{n+1}(q^{-1}x - q^{-1}|q)}{1+(q-1)x} = C_n(qx+1|q) \qquad (n \geq 1), \tag{2.10}$$

with the initial condition $C_1(x|q) = 1$ and $C_n(-\frac{1}{q}|q) = 0$ for $n \geq 2$.

For example, we have

$$\begin{aligned}
C_2(x|q) &= 1 + q\binom{x}{1}_q,\\
C_3(x|q) &= (1+q) + (q+q^2+q^3)\binom{x}{1}_q + q^4\binom{x}{2}_q,\\
C_4(x|q) &= (q^3+q^2+2q+1) + (q^6+q^5+2q^4+2q^3+2q^2+q)\binom{x}{1}_q\\
&\qquad + (q^9+q^8+q^7+q^6+q^5)q^{-1}\binom{x}{2}_q + q^9\binom{x}{3}_q.
\end{aligned}$$

It is clear that $C_n(x|q)$ is a polynomial in $\mathbb{Q}(q)[x]$ of degree $n-1$ for $n \geq 1$.

3. Some preliminary results

We first show that the evaluation of the polynomials $C_n(x|q)$ at q-integers is always a polynomial in $\mathbb{N}[q]$. Note that formulae (3.1) and (3.4) were implicitly given in [Cig97].

Proposition 2. *When $x = [k]_q$ we have*

$$C_{n+1}([k]_q|q) = q^{kn+\frac{n(n+1)}{2}} f(k+n, n|q^{-1}) \qquad (n,\, k \geq 0). \tag{3.1}$$

Proof. When $x = [k]_q$ Eq. (2.10) becomes

$$C_{n+1}([k]_q|q) = q^k C_n([k+1]_q|q) + C_{n+1}([k-1]_q|q). \tag{3.2}$$

It is easy to see that (3.1) is equivalent to (2.4). □

Corollary 3. *We have*

$$C_{n+1}(0|q) = C_n(1|q) \quad \textit{and} \quad C_{n+1}(1|q) = \widetilde{C}_{n+1}(q). \tag{3.3}$$

Proof. Letting $x = 0$ in (2.10) we get $C_{n+1}(0|q) = C_n(1|q)$. Letting $k = 1$ in (3.1) we have

$$\begin{aligned}
C_{n+1}(1|q) &= q^{n+\frac{n(n+1)}{2}} f(1+n, n|q^{-1})\\
&= q^{n+1+\frac{n(n+1)}{2}} f(n+1, n+1|q^{-1})\\
&= q^{\frac{n(n+1)}{2}} C_{n+1}(q^{-1}),
\end{aligned}$$

which is equal to $\widetilde{C}_{n+1}(q)$ by definition. □

The shifted factorial is defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1), \quad n = 1, 2, 3, \dots,$$

and $(x)_{-n} = 1/(x-n)_n$.

Proposition 4. *When $q = 1$ we have the explicit formula*

$$C_{n+1}(x|1) = \frac{(x+1)(x+n+2)_{n-1}}{n!} = \frac{x+1}{x+1+n}\binom{x+2n}{n} \quad (n \geq 0). \tag{3.4}$$

Proof. When $q = 1$ the equation (2.10) reduces to

$$C_{n+1}(x|1) = C_{n+1}(x-1|1) + C_n(x+1|1). \tag{3.5}$$

Since $C_{n+1}(x|1)$ is a polynomial in x of degree n, it suffices to prove that the right-hand side of (3.4) satisfy (3.5) for x being positive integers k. By Proposition 2 and (2.2) it suffices to check the following identity

$$\frac{k+1}{k+1+n}\binom{k+2n}{n} = \frac{k}{k+n}\binom{k-1+2n}{n} + \frac{k+2}{k+1+n}\binom{k+2n-1}{n-1}. \tag{3.6}$$

This is straightforward. □

To motivate our result in the next section we first prove two q-versions of a folklore result on the polynomials which take integral values on integers (see [St86, p. 38]). Introduce the polynomials $p_k(x)$ by

$$p_0(x) = 1 \quad \text{and} \quad p_k(x) = (-1)^k q^{-\binom{k}{2}} \frac{(x-1)(x-q)\cdots(x-q^{k-1})}{(q;q)_k}, \quad k \geq 1.$$

So $p_k(q^n) = {n \brack k}_q$ for $n \in \mathbb{N}$.

Proposition 5. *The following statements hold true.*

(i) *The polynomial $f(x)$ of degree k assumes values in $\mathbb{Z}[q]$ at $x = 1, q, \dots, q^k$ if and only if*

$$f(x) = c_0 + c_1 p_1(x) + \cdots + c_k p_k(x), \tag{3.7}$$

where $c_j = q^{\binom{j}{2}}(1-q)^j \mathcal{D}_q^j f(1)$ are polynomials in $\mathbb{Z}[q]$ for $0 \leq j \leq k$.

(ii) *The polynomial $\tilde{f}(x)$ of degree k assumes values in $\mathbb{Z}[q]$ at $x = 0, [1]_q, \dots, [k]_q$ if and only if*

$$\tilde{f}(x) = \sum_{j=0}^{k} \tilde{c}_j q^{-\binom{j}{2}} \binom{x}{j}_q, \tag{3.8}$$

where $\tilde{c}_j = q^{\binom{j}{2}} \Delta_q^j \tilde{f}(0)$ are polynomials in $\mathbb{Z}[q]$ for $0 \leq j \leq k$.

Proof. Clearly we can expand any polynomial $f(x)$ of degree k in the basis $\{p_j(x)\}_{0\leq j\leq k}$ as in (3.7). Besides, it is easy to see that

$$\mathcal{D}_q p_k(x) = \frac{q^{1-k}}{1-q} p_{k-1}(x) \Longrightarrow \mathcal{D}_q^j p_k(x) = \frac{q^{\binom{j+1}{2}-jk}}{(1-q)^j} p_{k-j}(x). \tag{3.9}$$

Hence, applying $\mathcal{D}_q^j$ to the two sides of (3.7) we obtain

$$\mathcal{D}_q^j f(1) = c_j \frac{q^{-\binom{j}{2}}}{(1-q)j} \Longrightarrow c_j = q^{\binom{j}{2}}(1-q)^j \mathcal{D}_q^j f(1).$$

Since $\mathcal{D}_q^j f(1)$ involves only the values of $f(x)$ at $x = 0, [1], \dots, [k]_q$ for $0 \leq j \leq k$, the result follows. In the same manner, since

$$\Delta_q \binom{x}{k}_q = \binom{x}{k-1}_q,$$

we obtain the expansion (3.8). □

Remark. (1) We can also derive (3.8) from (3.7) as follows. Let $y = \frac{x-1}{q-1}$. For any polynomial $f(x)$ define $\tilde{f}(y) = f(1+(q-1)y)$. Since $q^n = 1+(q-1)[n]_q$, it is clear that $f(q^n) \in \mathbb{Z}[q] \Longleftrightarrow \tilde{f}([n]_q) \in \mathbb{Z}[q]$. Writing $1 + qx - [j]_q = q(x - [j-1]_q)$ we see that

$$\tilde{f}_j(x) = q^{-\binom{j}{2}} \binom{x}{j}_q.$$

The expansion (3.8) follows from (3.7) immediately.

(2) When $\tilde{f}(x) = x^n$, it is known (see, for example, [Ze06]) that

$$\tilde{c}_k = \Delta_q 0^n = [k]_q! S_q(n,k),$$

where $[k]_q! = [1]_q \cdots [k]_q$ and $S_q(n,k)$ are classical q-Stirling numbers of the second kind defined by

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q S_q(n-1,k) \qquad \text{for} \quad n \geq k \geq 1,$$

with $S_q(n,0) = S_q(0,k) = 0$ except $S_q(0,0) = 1$.

(3) The two formulas (3.7) and (3.8) are special cases of the Newton interpolation formula, namely, for any polynomial f of degree less than or equal to n one has

$$f(x) = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \frac{f(b_j)}{\prod_{r=0, r\neq j}^{k} (b_j - b_r)} \right) (x-b_0)\cdots(x-b_{k-1}), \tag{3.10}$$

where $b_0, b_1, \dots, b_{n-1}$ are distinct complex numbers. Some recent applications of (3.10) in the computation of moments of Askey-Wilson polynomials are given in [GITZ].

4. Main results

In the light of Propositions 2 and 5, it is natural to consider the expansion of $C_{n+1}(x|q)$ and $C_{n+1}(qx+1|q)$ in the basis $\binom{x}{j}_q$ $(j \geq 0)$. It turns out that the coefficients in such expansions are Carlitz-Riordan's q-ballot numbers. Note that formula (4.2) was implicitly given in [Cig97].

Theorem 6. *For $n \geq 0$ we have*

$$C_{n+1}(x|q) = \sum_{j=0}^{n} f(n+j, n-j|q^{-1}) q^{jn+\frac{1}{2}(n-j)(n+j+1)} \binom{x}{j}_q, \tag{4.1}$$

$$C_n(qx+1|q) = \sum_{j=0}^{n-1} f(n+j, n-1-j|q^{-1}) q^{jn+\frac{1}{2}n(n+1)-\frac{1}{2}(j+1)(j+2)} \binom{x}{j}_q. \tag{4.2}$$

Proof. It is sufficient to prove the theorem for $x = [k]_q$ with $k = 0, 1, \dots, n$. By Proposition 2, the two equations (4.1) and (4.2) are equivalent to

$$f(k+n, n|q^{-1}) = \sum_{j=0}^{k} f(n+j, n-j|q^{-1}) q^{jn-kn-j} \begin{bmatrix} k \\ j \end{bmatrix}_q, \tag{4.3}$$

$$f(k+n, n-1|q^{-1}) = \sum_{j=0}^{k} f(n+j, n-j-1|q^{-1}) q^{jn-kn-2j+k} \begin{bmatrix} k \\ j \end{bmatrix}_q, \tag{4.4}$$

for $n \geq k \geq j$. Replacing q by $1/q$ and using $\begin{bmatrix} k \\ j \end{bmatrix}_{q^{-1}} = q^{-j(k-j)} \begin{bmatrix} k \\ j \end{bmatrix}_q$ we get

$$f(n+k, n|q) = \sum_{j=0}^{k} f(n+j, n-j|q) q^{(n-j)(k-j)+j} \begin{bmatrix} k \\ j \end{bmatrix}_q, \tag{4.5}$$

$$f(k+n, n-1|q) = \sum_{j=0}^{k} f(n+j, n-j-1|q) q^{(n-j-1)(k-j)+j} \begin{bmatrix} k \\ j \end{bmatrix}_q. \tag{4.6}$$

We only prove (4.5). By definition, the left-hand side $f(n+k, n|q)$ is the enumerative polynomial of lattice paths from $(0,0)$ to $(n+k+1, n)$ with $(1,0)$ as the last step. Each such path γ must cross the line $y = -x + 2n$. Suppose it crosses this line at the point $(n+j, n-j)$, $0 \leq j \leq k$. Then the path corresponds to a unique pair (γ_1, γ_2), where γ_1 is a path from $(0,0)$ to $(n+j, n-j)$ and γ_2 is a path from $(n+j, n-j)$

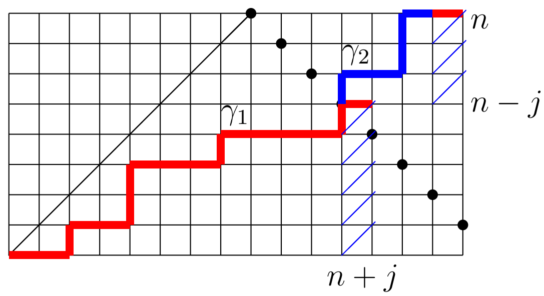

FIGURE 1. The decomposition $\gamma \mapsto (\gamma_1, \gamma_2)$

to $(n+k, n)$. It is clear that the area under the path γ is equal to $S_1 + S_2 + S_3 + j$, where

- S_1 is the area under the path γ_1', which is obtained from γ_1 plus the last step $(n+j, n-j) \to (n+j+1, n-j)$;
- S_2 is the area under the path γ_2 and above the line $y = n-j$;
- S_3 is the area of the rectangle delimited by the four lines $y = 0$, $y = n-j$, $x = n+j+1$ and $x = n+k+1$, i.e., $(n-j)(k-j)$.

This decomposition is depicted in Figure 1. Clearly, summing over all such lattice paths gives the summand on the right-hand side of (4.5). This completes the proof. $\square$

Remark. When $q = 1$, by (2.2), the above theorem implies that

$$\frac{x+1}{x+n+1}\binom{x+2n}{n} = \sum_{j=0}^{n} \frac{2j+1}{n+j+1}\binom{2n}{n-j}\binom{x}{j}, \tag{4.7}$$

$$\frac{x+2}{x+n+1}\binom{x+2n-1}{n-1} = \sum_{j=0}^{n-1} \frac{2j+2}{n+j+1}\binom{2n-1}{n-j-1}\binom{x}{j}. \tag{4.8}$$

Note that the two sequences

$$\{f(n+j, n-j|1)\} \quad \text{and} \quad \{f(n+j, n-j-1|1)\} \qquad (0 \le j \le n)$$

correspond, respectively, to the $(2n-1)$-th and $2n$-th anti-diagonal coefficients of the triangle $\{f(n,k)\}_{0 \le k \le n}$, see Table 1.

Theorem 7. *The polynomials $C_n(x|q)$ satisfy $C_1(x|q) = 1$ and*

$$[n]_q C_{n+1}(x|q) = ([2n-1]_q + xq^{2n-1})C_n(x|q) + \sum_{j=0}^{n-2}[n-j-1]_q \widetilde{C}_j(q) C_{n-j}(x|q) q^{2j+1}. \tag{4.9}$$

Proof. Since $C_{n+1}(x|q)$ is a polynomial in x of degree n, it suffices to prove (4.9) for $x = [k]_q$, where k is any positive integer, namely,

$$[n]_q C_{n+1}([k]_q|q) = [2n+k-1]_q C_n([k]_q|q) + \sum_{j=0}^{n-2}[n-j-1]_q \widetilde{C}_j(q) C_{n-j}([k]_q|q) q^{2j+1}. \tag{4.10}$$

Let $m \geq n$ and

$$\tilde{f}(m,n|q) = q^{(m-n)n+\binom{n+1}{2}} f(m,n|q^{-1}). \tag{4.11}$$

In view of the definition (2.3) it is clear that

$$\tilde{f}(m,n|q) = \sum_{\gamma \in \mathcal{P}(m,n)} q^{A'(\gamma)},$$

where $A'(\gamma)$ denotes the area above the path γ and under the line $y = x$ and $y = n$. Since

$$\widetilde{C}_j(q) = q^{\binom{j}{2}} C_j(q^{-1}) = q^{\binom{j+1}{2}} f(j,j|q^{-1}) = \tilde{f}(j,j|q),$$

using Proposition 2 and (4.11) with $m = k+n$, we can rewrite (4.10) as

$$[n]_q \tilde{f}(m,n|q) = [n+m-1]_q \tilde{f}(m-1,n-1|q) + \sum_{j=0}^{n-2} q^j [n-j-1]_q \tilde{f}(j,j|q) q^{j+1} [n-j-1]_q \tilde{f}(m-j-1,n-j-1|q). \tag{4.12}$$

A pointed lattice path is a pair (α,γ) such that $\alpha \in \{(1,1),\ldots,(n,n)\}$ and $\gamma \in \mathcal{P}(m,n)$. If $\alpha = (i,i)$ we call i the height of α and write $h(\alpha) = i$. Let $\mathcal{P}^*(m,n)$ be the set of all such pointed lattice paths. It is clear that the left-hand side of (4.12) has the following interpretation

$$[n]_q \tilde{f}(m,n|q) = \sum_{(\alpha,\gamma)\in\mathcal{P}^*(m,n)} q^{h(\alpha)-1+A'(\gamma)}. \tag{4.13}$$

Now, we compute the above enumerative polynomial of $\mathcal{P}^*(m,n)$ in another way in order to obtain the right-hand side of (4.12). We distinguish two cases.

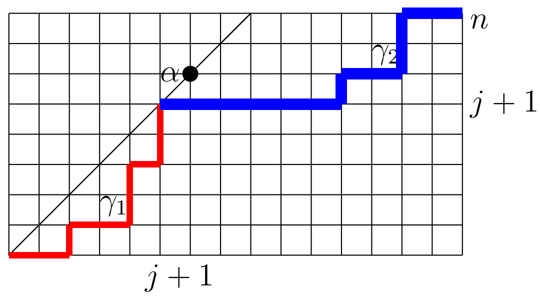

FIGURE 2. $(\alpha,\gamma)\to(\gamma_1,(\alpha',\gamma_2))$

- Let $\mathcal{P}_1^*(m,n,j)$ be the set of all pointed lattice paths (α,γ) in $\mathcal{P}^*(m,n)$ such that $h(\alpha)\in\{j+2,\dots,n\}$, where j is the smallest integer such that $(j+1,j)\to(j+1,j+1)$ is a step of γ, i.e., the first step of γ touching the line $y=x$. If $(\alpha,\gamma)\in\mathcal{P}_1^*(m,n,j)$, then we have the correspondence $(\alpha,\gamma)\to(\gamma_1,(\alpha',\gamma_2))$, where γ_1 is a lattice path from $(0,0)$ to $(j+1,j+1)$ which touches the line $y=x$ only at the two extremities, and (α',γ_2) is a pointed lattice path from $(0,0)$ to $(m-j,n-j-1)$ with $h(\alpha')=h(\alpha)-j-1$. This decomposition is depicted in Figure 2.

  Thus the corresponding enumerative polynomial of such paths for the fixed j is

$$\sum_{(\alpha,\gamma)\in\mathcal{P}_1^*(m,n,j)} q^{h(\alpha)-1+A(\gamma)} = q^j\tilde{f}(j,j|q)\cdot q^{j+1}[n-j-1]_q\tilde{f}(m-j-1,n-j-1|q).$$

  Summing over all j $(0\le j\le n-2)$ we obtain the second term on the right-hand side of (4.12).
- Let $\mathcal{P}_2^*(m,n)$ be the set of all pointed lattice paths (α,γ) in $\mathcal{P}^*(m,n)$ such that $h(\alpha)\in\{1,\dots,n\}$ and $h(\alpha)\le j+1$ where j (if any) is the smallest integer such that $(j+1,j)\to(j+1,j+1)$ is a step of γ, i.e., the first step of γ touching the line $y=x$. If $(\alpha,\gamma)\in\mathcal{P}_2^*(m,n)$, where $\gamma=(p_0,\dots,p_{m+n+1})$ with $p_0=(0,0)$ and $p_{m+n+1}=(m+n+1,n)$, we can associate a pair (i,γ') where $\gamma'\in\mathcal{P}(m-1,n-1)$ is obtained from γ by deleting the vertical step $(x,h(\alpha)-1)\to(x,h(\alpha))$ and the first horizontal step $(0,0)\to(1,0)$, i.e.,

$$\gamma'=(p'_1,\dots,p'_i,p'_{i+2},\dots,p'_{n+m+1})$$

  where $i=x+h(\alpha)-1$, $p'_k=p_k-(1,0)$ if $k=1,\dots,i$ and $p'_k=p_k-(0,1)$ if $k=i+2,\dots,m+n+1$. It is easy to see that the mapping $(\alpha,\gamma)\mapsto(i,\gamma')$ is a bijection, which is depicted in Figure 3.

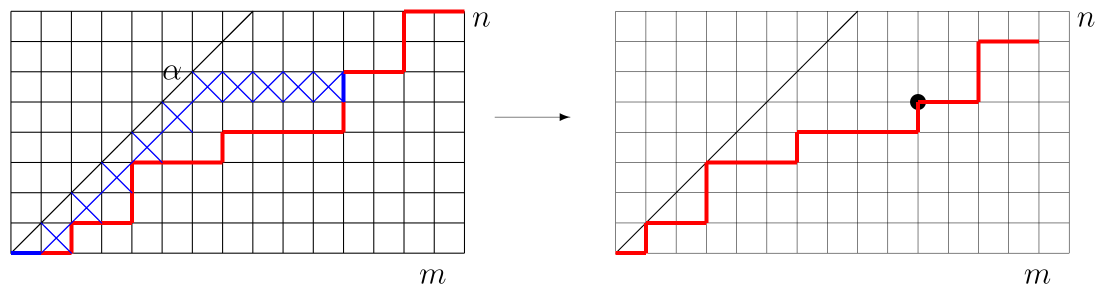

FIGURE 3. $(\alpha,\gamma)\mapsto(i,\gamma')$ with $m=15$, $n=8$, $\alpha=(6,6)$ and $i=15$

Since $1\le x\le m$ and $0\le h(\alpha)-1\le n-1$ we have $i\in\{1,\dots,m+n-1\}$. As $A'(\gamma)=x-1+A'(\gamma')$ we have

$$h(\alpha)-1+A'(\gamma)=i-1+A'(\gamma').$$

It follows that

$$\sum_{(\alpha,\gamma)\in\mathcal{P}_2^*(m,n)} q^{h(\alpha)+A'(\gamma)}=\sum_{i=1}^{m+n-1} q^{i-1}\sum_{\gamma'} q^{A'(\gamma')}=[n+m-1]_q\tilde{f}(m-1,n-1|q).$$

Summing up the two cases we obtain the right-hand side of (4.12). □

When $q=1$ we have an alternative proof of Theorem 7.

Another proof of the $q=1$ case. When $q=1$ Eq. (4.9) reduces to

$$nC_{n+1}(x|1)=(2n-1+x)C_n(x|1)+\sum_{j=0}^{n-2}(n-j-1)C_jC_{n-j}(x|1)\quad (n\ge 2). \tag{4.14}$$

This yields immediately $C_1(x|1)=1$, $C_2(x|1)=x+1$, $C_3(x|1)=(x+1)(x+4)/2$, in accordance with the formula (3.4). For $n\ge 3$, letting $k=n-j-3$, $N=n-3$ and $z=x+3$, by (3.4), the recurrence (4.14) is equivalent to the following identity

$$\frac{(z+N+2)_N}{N!}=\sum_{k=0}^{N}4^{N-k}\frac{(3/2)_{N-k}(z+k)_k}{(3)_{N-k}k!}\qquad (N\ge 0).$$

Notice that we can rewrite the right-hand side as

$$\begin{aligned}
&\frac{(3/2)_N}{(3)_N}4^N\sum_{k=0}^{N}\frac{(-2-N)_k((z+1)/2)_k(z/2)_k}{k!(-1/2-N)_k(z)_k}\\
=&\frac{(3/2)_N}{(3)_N}4^N\left({}_3F_2\left(\begin{matrix}-2-N, & (z+1)/2, & z/2\\ -1/2-N, & z, & \end{matrix};1\right)\right.\\
&\qquad-\frac{(-2-N)_{N+1}((z+1)/2)_{N+1}(z/2)_{N+1}}{(-1/2-N)_{N+1}(z)_{N+1}(N+1)!}\\
&\qquad\left.-\frac{(-2-N)_{N+2}((z+1)/2)_{N+2}(z/2)_{N+2}}{(-1/2-N)_{N+2}(z)_{N+2}(N+2)!}\right).
\end{aligned}$$

Invoking Pfaff-Saalschütz formula [AAR99, Theorem 2.2.6] we obtain

$$ {}_3F_2\left(\begin{matrix}-2-N, & (z+1)/2, & z/2\\ -1/2-N, & z, & \end{matrix};1\right)=\frac{(z/2)_{N+2}((z-1)/2)_{N+2}}{(z)_{N+2}(-1/2)_{N+2}}.$$

Substituting this in the previous expression yields $\frac{(z+N+2)_N}{N!}$ after simplification. $\square$

When $x=1$ Eq. (4.14) reduced to the following identity for Catalan numbers:

$$nC_{n+1}=2nC_n+\sum_{j=0}^{n-2}(n-j-1)C_jC_{n-j}. \tag{4.15}$$

5. Concluding remarks

We conclude this paper with a few open problems. By Theorem 6 it is clear that $C_{n+1}(x|q)$ is a polynomial in x of degree n with leading coefficient $q^{n^2}/[1]_q[2]_q\cdots[n]_q$.

Conjecture 8. *One can write $C_n(x|q)$ as an irreducible fraction*

$$\frac{P_n(x|q)}{[1]_q\cdots[n-1]_q},$$

where P_n has only positive coefficients.

This has been checked up to $n=27$. The similar conjecture is true when $q=1$ by Proposition 4.

Finally, the Newton polytope of the numerator of $C_n(x|q)$ seems to have a nice shape. This is illustrated in Figure 4, where the horizontal axis is associated with powers of q and the vertical axis with powers of x. The slopes of the upper part seems to be given in general by the odd integers 1, 3, …, $2n-3$.

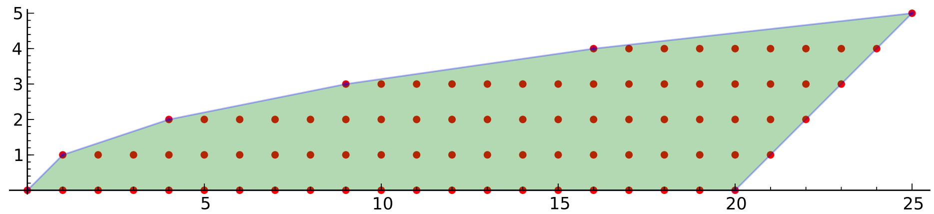

FIGURE 4. Newton polytope of the numerator of $C_n(x|q)$ for $n = 6$

REFERENCES

[And75] G. Andrews, Identities in combinatorics. II. A q-analog of the Lagrange inversion theorem. Proc. Amer. Math. Soc. 53 (1975), no. 1, 240–245.

[AAR99] G. Andrews, R. Askey, R. Roy, Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.

[BP12] S. A. Blanco and T. Kyle Petersen, Counting Dyck paths by area and rank, arXiv:1206.0803v1.

[BF07] L. M. Butler and W. P. Flanigan, A note on log-convexity of q-Catalan numbers, Ann. Comb. 11 (2007), no. 3-4, 369–373.

[CR64] L. Carlitz and J. Riordan, Two element lattice permutation numbers and their q-generalization, Duke Math. J., 31:371–388, 1964.

[Car72] L. Carlitz, Sequences, paths, ballot numbers, Fibonacci Quart. 10 (1972), no. 5, 531–549.

[Cig97] J. Cigler, Operatormethoden für q-Identitäten. IV. Eine Klasse von q-Gould-Polynomen. Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 205 (1996), 169–174 (1997).

[Cig98] J. Cigler, Operatormethoden für q-Identitten. VI. Geordnete Wurzelbäume und q-Catalan-Zahlen, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 206 (1997), 253–266 (1998).

[Cig05] J. Cigler, q-Catalan numbers and q-Narayana polynomials, arXiv:math/0507225.

[Cha13a] F. Chapoton, Sur une série en arbres à deux paramètres, Séminaire Lotharingien de Combinatoire vol. 70 (2013)

[Cha13b] F. Chapoton, Flows on rooted trees and the Menous-Novelli-Thibon idempotents, to appear in Mathematica Scandinavica (2013).

[Com74] L. Comtet, Advanced Combinatorics, D. Reidel, Dordrecht, Holland, 1974.

[FH85] J. Fürlinger, J. Hofbauer, q-Catalan numbers J. Combin. Theory Ser. A, 40 (1985), pp. 248–264

[GITZ] V. J..W Guo, M. Ishikawa, H. Tagawa, J. Zeng, A quadratic formula for basic hypergeometric series related to Askey-Wilson polynomials, to appear in Proc. Amer. Math. Soc.

[Hag08] J. Haglund, The q,t-Catalan numbers and the space of diagonal harmonics. With an appendix on the combinatorics of Macdonald polynomials. University Lecture Series, 41. American Mathematical Society, Providence, RI, 2008.

[Rein05] M. Reineke, Cohomology of noncommutative Hilbert schemes, Algebr. Represent. Theory 8 (2005), no. 4, 541–561.

[St86] R. Stanley, Enumerative combinatorics. Vol. I. With a foreword by Gian-Carlo Rota. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986.

[Ze06] J. Zeng, The Akiyama–Tanigawa algorithm for Carlitz's q-Bernoulli numbers, Integers 6 (2006) A5.

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